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INTERPOLATION IN FRAGMENTS OF INTUITIONISTIC PROPOSITIONAL LOGIC

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Abstract. We show in this paper that all fragments of intuitionistic propositional logic based on a subset of the connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$ satisfy interpolation. Fragments containing $\leftrightarrow$ or $\neg\neg$ are briefly considered.

§1. Introduction.

1.1. A *fragment* $[C]$ of IpL (intuitionistic propositional logic) is a subset of the set of formulae of IpL built up from the propositional variables and constants ($\top$ and $\bot$) by means of connectives from the set C only. If $C = \{c_1, c_2, \dots\}$, then we write $[c_1, c_2, \dots]$ for $[C]$. In this note, we mainly consider the *primitive* connectives $\wedge$, $\vee$ and $\rightarrow$; one can, however, also think of *defined* connectives c_A where $A = A(P_1, \dots, P_n)$ is some formula, with $c_A(B_1, \dots, B_n) := A(B_1, \dots, B_n)$. Examples of defined connectives are $\neg$ ($\neg A = A \rightarrow \bot$) and $\leftrightarrow$ ($A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A)$). So e.g. $[\wedge, \vee, \rightarrow]$ contains all formulae of IpL, and $[\leftrightarrow]$ is the fragment containing all formulae built up with $\leftrightarrow$ only.

1.2. The *interpolation theorem* for IpL reads as follows:

Let A and B be formulae of IpL such that $A \vdash B$. Then there is an interpolant I for $A \vdash B$, i.e.

i) *$A \vdash I$ and $I \vdash B$; and*

ii) *all propositional variables of I occur both in A and in B.*

This theorem is a consequence of the interpolation theorem for intuitionistic predicate logic, first proved by Schütte in [S62].

1.3. In this paper, we consider relativizations of the interpolation theorem to *elementary* fragments (fragments based on primitive connectives or $\neg$), and we show

interpolation holds in all elementary fragments.

There are many fragments of intuitionistic propositional logic for which interpolation fails, e.g. $[\wedge, \rightarrow, \neg, \delta]$ with $\delta(A, B, C) = (A \vee \neg A) \wedge (A \rightarrow B) \wedge (\neg A \rightarrow C)$

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(first proved by J. Zucker in [Z78]; see also [R81]). For intermediate logics the situation is the same (see [P85]). In classical logic, however, interpolation holds in all propositional fragments (proved by F. Ville: see [KK71] or [KK72, the exercises to Chapter 1]).

Another notion of fragment is considered in [R86], where a strong version of interpolation is proved for the subset NNIL (No Nestings of Implication to the Left) of formulae of IpL, defined inductively as follows: all propositional variables and constants are in NNIL; NNIL is closed under $\wedge$ and $\vee$; and if $A \in$ NNIL and P is a propositional variable, then $P \to A \in$ NNIL.

1.4. The rest of this paper is organized as follows: in §2 we fix the notation and present a sequent calculus for IpL, §3 consists of three lemmata about elementary fragments, §4 contains Schütte's method to prove the interpolation theorem for IpL, which is used subsequently to show interpolation for all elementary fragments, and in §5 we discuss the consequences of not adding the constants $\top$ and $\bot$ to the fragments. §6 is rather tentative: it reports on unsuccessful attempts to prove interpolation for some fragments containing the connectives $\leftrightarrow$ and $\neg\neg$.

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§2. Preliminaries.

2.1. *Notation.* All formulae are in intuitionistic propositional logic, with $\wedge$, $\vee$, $\to$, $\neg$ as connectives and the constants $\top$ and $\bot$. $P, Q, \ldots$ are propositional variables; together with $\top$ and $\bot$ we call them *atoms*. $A, B, C, \ldots$ are formulae; $\Gamma, \Delta, \Gamma', \ldots$ are finite (possibly empty) sets of formulae. We write Γ, Δ for the union of Γ and Δ; Γ, A stands for $\Gamma, \{A\}$.

For sets of formulae F, G we define

$$F \equiv G := \forall A \in F \exists B \in G (A \equiv B) \text{ and } \forall B \in G \exists A \in F (B \equiv A),$$

where $A \equiv B$ stands for $A \vdash B$ *and* $B \vdash A$. We also put

$$\begin{aligned} \bigwedge F &:= \{A_1 \wedge \cdots \wedge A_n \mid A_1, \ldots, A_n \in F\}, \\ F \to G &:= \{A \to B \mid A \in F, B \in G\}, \\ \neg F &:= \{\neg A \mid A \in F\}. \end{aligned}$$

We define $a(A)$ $[a^+(A)]$, the set of all [strictly positively occurring] atoms in A, by

$$\begin{aligned} a(\top) &= a^+(\top) = \{\top\}, \\ a(\bot) &= a^+(\bot) = \{\bot\}, \\ a(P) &= a^+(P) = \{P\}, \\ a(A \wedge B) &= a(A \vee B) = a(A \to B) = a(A) \cup a(B), \\ a(\neg A) &= a(A), \\ a^+(A \wedge B) &= a^+(A \vee B) = a^+(A) \cup a^+(B), \\ a^+(A \to B) &= a^+(B), \\ a^+(\neg A) &= \varnothing; \end{aligned}$$

$p(A)$ $[p^+(A)]$, the set of all [strictly positively occurring] propositional variables in

A, is defined by

$$p(A) = a(A) - \{\top, \bot\}, \qquad p^+(A) = a^+(A) - \{\top, \bot\}.$$

2.2. *The derivation system.* We use the following sequent calculus, denoted by $\mathbb{SC}$:

(P) $\quad \Gamma, P \vdash P,$

$(\top)$ $\quad \Gamma \vdash \top,$

$(\bot)$ $\quad \Gamma, \bot \vdash C,$

$$(\wedge\mathrm{R})\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}, \qquad (\wedge\mathrm{L})\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C},$$

$$(\vee\mathrm{R})\ \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (i = 1, 2), \qquad (\vee\mathrm{L})\ \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C},$$

$$(\to\mathrm{R})\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}, \qquad (\to\mathrm{L})\ \frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C},$$

$$(\neg\mathrm{R})\ \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}, \qquad (\neg\mathrm{L})\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B}.$$

The *main formula* of a rule is the newly formed formula of the conclusion: $A \wedge B$ for $(\wedge\mathrm{R})$ and $(\wedge\mathrm{L})$, $A_1 \vee A_2$ for $(\vee\mathrm{R})$, $A \vee B$ for $(\vee\mathrm{L})$, $A \to B$ for $(\to\mathrm{R})$ and $(\to\mathrm{L})$, $\neg A$ for $(\neg\mathrm{R})$ and $(\neg\mathrm{L})$.

$\mathbb{SC}$ has the following derived rules:

(CUT) Cut Elimination. If $\Gamma \vdash A$ and $\Gamma, A \vdash B$ then $\Gamma \vdash B$.

(WEAK) Weakening. If $\Gamma \vdash A$ then $\Gamma, \Delta \vdash A$.

(SUB) Substitution. If $\Gamma \vdash A$ then $\Gamma[P := B] \vdash A[P := B]$.

The proofs are standard (as for related systems, e.g. in [S62] and [T75]).

Note that the subformula property only holds in the following version:

if B occurs in a cut-free derivation of $\Gamma \vdash A$,
then $B = \bot$ or B is a subformula of Γ, A;

the addition $B = \bot$ is made necessary by the presence of $(\neg\mathrm{R})$, in which $\bot$ is eliminated. The following consequence is important in the context of this paper:

Let Π be a derivation of $\Gamma \vdash A$. Then we have:

i) *if B is a formula occurring in Π, then all connectives in B occur in Γ, A; and*

ii) *if $c \in \{\wedge, \vee, \to, \neg\}$ and* (cR) *or* (cL) *is a rule applied in Π, then c occurs in Γ, A.*

For later use (4.5, 4.6), we define a variant $\mathbb{SC}^*$ of $\mathbb{SC}$ and prove it equivalent to $\mathbb{SC}$.

2.3. $\mathbb{SC}^*$ is $\mathbb{SC}$ with $(\vee\mathrm{L})$ and $(\to\mathrm{L})$ replaced by:

$$(\vee\mathrm{L})^*\ \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ (A \vee B \notin \Gamma),$$

$$(\to\mathrm{L})^*\ \frac{\Gamma, A \to B \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ (A \to B \notin \Gamma).$$

We write $\Gamma \vdash^* A$ for: $\Gamma \vdash A$ is derivable in $\mathbb{S}\mathbf{C}^*$.

2.4. LEMMA. *$\Gamma \vdash A$ if and only if $\Gamma \vdash^* A$.*

PROOF. We write $\Gamma \vdash_n A$ for "$\Gamma \vdash A$ has a derivation with length at most n"; idem for $\Gamma \vdash_n^* A$. By induction over n one easily proves:

(1) if $\Gamma, A \vee B, A \vdash_n C$, then $\Gamma, A \vdash_n C$;

(2) if $\Gamma, A \vee B, B \vdash_n C$, then $\Gamma, B \vdash_n C$;

(3) if $\Gamma, A \to B, B \vdash_n C$, then $\Gamma, B \vdash_n C$;

(4) if $\Gamma \vdash_n C$, then $\Gamma, \Delta \vdash_n C$;

(5) if $\Gamma \vdash_n^* C$, then $\Gamma, \Delta \vdash_n^* C$.

We turn to the "if" part of the lemma. Assume $\Gamma \vdash^* A$, i.e. $\Gamma \vdash_n^* A$ for some n; we show $\Gamma \vdash_n A$ by induction over n. If $n = 1$ then $\Gamma \vdash^* A$ is an axiom, hence $\Gamma \vdash A$; if $n > 1$ and $\Gamma \vdash^* A$ is (an axiom or) the conclusion of $(\vee \mathrm{L})^*$ or a rule different from $(\to\mathrm{L})^*$, then the result directly follows from the induction hypothesis (using that every instance of $(\vee \mathrm{L})^*$ is an instance of $(\vee \mathrm{L})$). If $\Gamma \vdash^* A$ is the conclusion of $(\to\mathrm{L})^*$, then the premises are of the form $\Gamma', B \to C \vdash^* B$ and $\Gamma', C \vdash^* A$ where $\Gamma' := \Gamma - \{B \to C\}$. Now apply (5) to obtain $\Gamma', B \to C, C \vdash_{n-1}^* A$, and then the induction hypothesis.

Finally we prove the "only if" part, by induction over the length of a derivation of $\Gamma \vdash A$. Assume $\Gamma \vdash A$, so $\Gamma \vdash_n A$ for some n. If $n = 1$ then $\Gamma \vdash A$ is an axiom, hence $\Gamma \vdash^* A$; if $n > 1$ and $\Gamma \vdash A$ is (an axiom or) the conclusion of an instance of $(\vee \mathrm{L})^*$ or a rule different from $(\vee \mathrm{L})$, $(\to\mathrm{L})$, then the result directly follows from the induction hypothesis. There are three cases left:

i) $\Gamma \vdash A$ is the conclusion of $(\vee \mathrm{L})$ with premises of the form $\Gamma', B \vee C, B \vdash A$ and $\Gamma', B \vee C, C \vdash A$ where $\Gamma' := \Gamma - \{B \vee C\}$: apply (1), (2), the induction hypothesis and $(\vee \mathrm{L})^*$.

ii) $\Gamma \vdash A$ is the conclusion of $(\to\mathrm{L})$ with premises of the form $\Gamma' \vdash B$ and $\Gamma', C \vdash A$ where $\Gamma' := \Gamma - \{B \to C\}$: apply (4) to obtain $\Gamma', B \to C \vdash_{n-1} B$, then the induction hypothesis and $(\to\mathrm{L})^*$.

iii) $\Gamma \vdash A$ is the conclusion of $(\to\mathrm{L})$ with premises of the form $\Gamma', B \to C \vdash B$ and $\Gamma', B \to C, C \vdash A$ where $\Gamma' := \Gamma - \{B \to C\}$: apply (3) to obtain $\Gamma', C \vdash_{n-1} A$, then the induction hypothesis and $(\to\mathrm{L})^*$. □

§3. Elementary fragments. Before turning to interpolation, we derive some properties of elementary fragments.

3.1. LEMMA. (i) $[\wedge, \to] \equiv \bigwedge[\to]$.

(ii) *Let $A \in [\to]$. Then $a^+(A) = \{X\}$ for some atom X, and $A \equiv (A \to X) \to X$.*

(iii) $\{A \in [\wedge, \to] \mid a^+(A)$ *is a singleton*$\} \equiv [\to]$.

PROOF. (i) Formula induction, using

$$(A \wedge B) \to (C \wedge D) \equiv (A \to (B \to C)) \wedge (A \to (B \to D)).$$

(ii) We have $A = A_1 \to (A_2 \to \cdots \to (A_n \to X) \cdots)$ $(n \geq 0)$ for some atom X, so $a^+(A) = \{X\}$. Also

$$A \equiv (A_1 \wedge \cdots \wedge A_n) \to X \equiv (((A_1 \wedge \cdots \wedge A_n) \to X) \to X) \to X \equiv (A \to X) \to X.$$

(iii) Formula induction, using (i), (ii) and

$$((A \to X) \to X) \wedge ((B \to X) \to X) \equiv (A \to (B \to X)) \to X. \qquad \square$$

3.2. LEMMA. (i) $[\wedge, \vee, \to] \equiv \bigwedge[\vee, \to]$.

(ii) *There is a mapping* $\Rightarrow$ *such that if* $A \in [\wedge, \vee, \to]$ *and* $B \in [\vee, \to]$, *then* $A \Rightarrow B \in [\vee, \to]$, $A \Rightarrow B \equiv (A \to B)$ *and* $p(A \Rightarrow B) = p(A \to B)$; *as a consequence we have* $[\wedge, \vee, \to] \to [\vee, \to] \equiv [\vee, \to]$.

PROOF. (i) Formula induction, using the following equivalences:

$$(A \wedge B) \vee (C \wedge D) \equiv (A \vee C) \wedge (B \vee C) \wedge (A \vee D) \wedge (B \vee D),$$
$$(A \wedge B) \to (C \wedge D) \equiv (A \to (B \to C)) \wedge (A \to (B \to D)).$$

(ii) Let $A \in [\wedge, \vee, \to]$ and $B \in [\vee, \to]$. By (i) we have $A \equiv A_1 \wedge \cdots \wedge A_n$ with $A_i \in [\vee, \to]$ $(i = 1, \ldots, n)$; from the proof of (i) it follows that $p(A) = p(A_1 \wedge \cdots \wedge A_n)$. Now put

$$A \Rightarrow B := A_1 \to (A_2 \to \cdots \to (A_n \to B) \cdots),$$

and one easily sees that (ii) is satisfied. $\square$

3.3. LEMMA. (i) $[\wedge, \vee, \neg] \equiv \bigwedge[\vee, \neg]$.

(ii) *There is a mapping* $\Downarrow$ *such that if* $A \in [\wedge, \vee, \neg]$, *then* $\Downarrow A \in [\vee, \neg]$, $\Downarrow A \equiv \neg A$ *and* $p(\Downarrow A) = p(\neg A)$; *as a consequence we have* $\neg[\wedge, \vee, \neg] \equiv [\vee, \neg]$.

PROOF. Analogous to that of 3.2, using the equivalence

$$\neg(A \wedge B) \equiv \neg\neg(\neg A \vee \neg B)$$

and the definition

$$\Downarrow A := \neg\neg(\neg A_1 \vee \cdots \vee \neg A_n)$$

for $A \in [\wedge, \vee, \neg]$ with $A \equiv A_1 \wedge \cdots \wedge A_n$ and $A_i \in [\vee, \neg]$ $(i = 1, \ldots, n)$. $\square$

§4. Interpolation in elementary fragments.

4.1. LEMMA (SCHÜTTE [S62]). *Interpolation holds in* $[\wedge, \vee, \to, \neg]$.

PROOF. Let $A \vdash B$. Then there is a derivation in $\mathbb{SC}$ of $A \vdash B$. With induction over the length of the derivation it is shown that any partition $\Gamma, \Delta \vdash C$ of a sequent in the derivation has an interpolant I, i.e.

$$\Gamma \vdash I; \qquad I, \Delta \vdash C; \qquad p(\Gamma) \cap p(\Delta, C) \supseteq p(I).$$

From this the lemma follows (take $\Gamma = \{A\}$, $\Delta = \varnothing$ and $C = B$).

The method to obtain the interpolant I for $\Gamma, \Delta \vdash A$ can be rendered as follows:

$$(\mathrm{i}P1) \quad \Gamma[\top]\Delta, P \vdash P, \qquad (\mathrm{i}P2) \quad \Gamma, P[P]\Delta \vdash P,$$
$$(\mathrm{i}\top) \quad \Gamma[\top]\Delta \vdash \top,$$
$$(\mathrm{i}\bot 1) \quad \Gamma[\top]\Delta, \bot \vdash C, \qquad (\mathrm{i}\bot 2) \quad \Gamma, \bot[\bot]\Delta \vdash C,$$
$$(\mathrm{i}\wedge\mathrm{R}) \quad \frac{\Gamma[I_1]\Delta \vdash A \qquad \Gamma[I_2]\Delta \vdash B}{\Gamma[I_1 \wedge I_2]\Delta \vdash A \wedge B},$$
$$(\mathrm{i}\vee\mathrm{L1}) \quad \frac{\Gamma[I_1]A, \Delta \vdash C \qquad \Gamma[I_2]B, \Delta \vdash C}{\Gamma[I_1 \wedge I_2]A \vee B, \Delta \vdash C},$$

$$(\mathrm{i} \vee \mathrm{L2}) \quad \frac{\Gamma, A[I_1]\Delta \vdash C \qquad \Gamma, B[I_2]\Delta \vdash C}{\Gamma, A \vee B[I_1 \vee I_2]\Delta \vdash C},$$

$$(\mathrm{i} \to \mathrm{L1}) \quad \frac{\Gamma[I_1]\Delta \vdash A \qquad \Gamma[I_2]B, \Delta \vdash C}{\Gamma[I_1 \wedge I_2]A \to B, \Delta \vdash C},$$

$$(\mathrm{i} \to \mathrm{L2}) \quad \frac{\Delta[I_1]\Gamma \vdash A \qquad \Gamma, B[I_2]\Delta \vdash C}{\Gamma, A \to B[I_1 \to I_2]\Delta \vdash C},$$

$$(\mathrm{i} \neg \mathrm{L1}) \quad \frac{\Gamma[I]\Delta \vdash A}{\Gamma[I]\Delta, \neg A \vdash B}, \qquad (\mathrm{i} \neg \mathrm{L2}) \quad \frac{\Delta[I]\Gamma \vdash A}{\Gamma, \neg A[\neg I]\Delta \vdash B}.$$

We explain this notation with an example. (i $\wedge$ R) means

$$\text{if } \Gamma \vdash I_1 \text{ and } I_1, \Delta \vdash A \text{ and } \Gamma \vdash I_2 \text{ and } I_2, \Delta \vdash B$$
$$\text{then } \Gamma \vdash I_1 \wedge I_2 \text{ and } I_1 \wedge I_2, \Delta \vdash A \wedge B;$$

so (i $\wedge$ R) indicates how an interpolant for Γ, $\Delta \vdash A \wedge B$ can be obtained from interpolants for Γ, $\Delta \vdash A$ and Γ, $\Delta \vdash B$.

For rules not mentioned here (($\wedge$ L), ($\vee$ R), ($\to$R) and ($\neg$ R)), the interpolant for the conclusion is the same as for the premise. □

4.2. With Schütte's method (i.e. the method used in the proof of Lemma 4.1), it is easy to prove interpolation for $[\neg]$ and for the fragments containing $\wedge$ ($[\wedge]$, $[\wedge, \vee]$, $[\wedge, \to]$, $[\wedge, \neg]$, $[\wedge, \vee, \to]$, $[\wedge, \vee, \neg]$, $[\wedge, \to, \neg]$, $[\wedge, \vee, \to, \neg]$).

This is not evident for fragments containing $\vee$ or $\to$, but not $\wedge$ ($[\vee]$, $[\to]$, $[\vee, \to]$, $[\vee, \neg]$, $[\to, \neg]$, $[\vee, \to, \neg]$), as (i $\vee$ L1) and (i $\to$ L1) introduce $\wedge$ in the definition of the interpolant. To illustrate this, we present the following example where Schütte's method is applied to a proof of $(P \vee Q) \to R \vdash (P \vee Q) \to R$:

$$\dfrac{\dfrac{\dfrac{\dfrac{P, (P \vee Q) \to R[P] \vdash P}{P, (P \vee Q) \to R[P] \vdash P \vee Q} \qquad R[R]P \vdash R}{(P \vee Q) \to R[P \to R]P \vdash R} \qquad \dfrac{\dfrac{Q, (P \vee Q) \to R[Q] \vdash Q}{Q, (P \vee Q) \to R[Q] \vdash P \vee Q} \qquad R[R]Q \vdash R}{(P \vee Q) \to R[Q \to R]Q \vdash R}}{(P \vee Q) \to R[(P \to R) \wedge (Q \to R)]P \vee Q \vdash R}}{(P \vee Q) \to R[(P \to R) \wedge (Q \to R)] \vdash (P \vee Q) \to R}$$

Since $\neg$ is definable in fragments containing $\to$ (using the constant $\bot$), we have $[\to, \neg] \equiv [\to]$ and $[\vee, \to, \neg] \equiv [\vee, \to]$, so the only fragments for which interpolation still has to be shown are $[\vee]$, $[\to]$, $[\vee, \to]$ and $[\vee, \neg]$. We take a closer look at these four fragments in the rest of this section.

4.3. Interpolation in $[\vee]$ is trivial: if $A, B \in [\vee]$ and $A \vdash B$, then $A = P_1 \vee \cdots \vee P_m$, $B = Q_1 \vee \cdots \vee Q_n$ with $\forall i \leq m \exists j \leq n (P_i = Q_j)$, so $p(A) \subseteq p(B)$ and A is an interpolant. (By a similar argument, interpolation in $[\wedge]$ is trivial.)

4.4. THEOREM (ZUCKER [Z78]). *Interpolation holds in* $[\to]$.

PROOF (somewhat different from Zucker's). Let $A, B \in [\to]$ with $A \vdash B$. Schütte's method gives us an interpolant I in the fragment $[\wedge, \to]$. By induction over the length of the derivation of $A \vdash B$ one can show that $p^+(I) \subseteq p^+(A)$, so by Lemma 3.1(iii) there is an I' equivalent to I in $[\to]$. □

4.5. THEOREM. *Interpolation holds in* $[\vee, \to]$.

PROOF. Let A, $B \in [\vee, \to]$ with $A \vdash B$. By Lemma 2.4 there is a derivation Π in $\mathfrak{S}\mathbf{C}^*$ of $A \vdash^* B$. We define inductively the subtree Π^+ of Π:

a) $A \vdash^* B$ is in Π^+; and

b) if the conclusion of a rule is in Π^+, then so is (are) the premise(s), with the following exception:

(*) if the rule involved is an instance of $(\to \mathrm{L})^*$ with a subformula occurrence of A as main formula, then only the right-hand premise is in Π^+.

By induction over the structure of Π^+ one easily observes (using the peculiar definition of $(\vee \mathrm{L})^*$ and $(\to \mathrm{L})^*$) that

(1) *every sequent in Π^+ is of the form $A_0, \Gamma \vdash^* C$, where A_0 is a strictly positive subformula occurrence of A and $\Gamma \cup \{C\}$ a collection of subformula occurences of B.*

In other words: the sequents in Π^+ contain no negative subformula occurrences of A and exactly one strictly positive subformula occurrence of A, namely in the premise.

Now we prove, by formula induction:

(2) *for every strictly positive subformula occurrence A_0 of A, there is an $I = I(A_0, \Pi^+)$ such that* (i) $I \in [\vee, \to]$, (ii) $p(I) \subseteq p(A) \cap p(B)$, (iii) $A_0 \vdash I$, *and* (iv) *for every sequent $A_0, \Gamma \vdash^* C$ ($A_0 \notin \Gamma$) in Π^+ we have $I, \Gamma \vdash C$;*

from this the theorem follows (take A for A_0).

(I) $A_0 = P$, *P an atom.* Take

$$I(P, \Pi^+) := \begin{cases} P & \text{if } P \in p(B), \\ \top & \text{otherwise;} \end{cases}$$

then (i), (ii) and (iii) are trivial. For (iv), we argue as follows: if $P \in p(B)$, then $I(P, \Pi^+) = P = A_0$ and (iv) holds trivially; if $P \notin p(B)$, then $P \notin p(\Gamma, C)$ by (1), and with $A_0, \Gamma \vdash C$ we get (using $A_0 = P$ and (SUB)) $\top, \Gamma \vdash C$.

(II) $A_0 = A_1 \vee A_2$. Now put

$$I(A_1 \vee A_2, \Pi^+) := I(A_1, \Pi^+) \vee I(A_2, \Pi^+);$$

(i), (ii) and (iii) are evident, and (iv) is proved as follows by induction over the length of a subderivation of Π^+ containing only sequents of the form $A_1 \vee A_2, \Gamma \vdash^* C$ ($A_1 \vee A_2 \notin \Gamma$). We distinguish three cases, writing I for $I(A_1 \vee A_2, \Pi^+)$.

(a) The sequent is an axiom; then so is $I, \Gamma \vdash C$.

(b) $A_1 \vee A_2, \Gamma \vdash^* C$ is the conclusion of a $(\vee \mathrm{L})^*$-rule:

$$\frac{A_1, \Gamma \vdash^* C \qquad A_2, \Gamma \vdash^* C}{A_1 \vee A_2, \Gamma \vdash^* C};$$

by the induction hypothesis (2), we have $I(A_1, \Pi^+), \Gamma \vdash C$ and $I(A_2, \Pi^+), \Gamma \vdash C$; hence we have $I, \Gamma \vdash C$ by $(\vee \mathrm{L})^*$.

(c) $A_1 \vee A_2$, $\Gamma \vdash^* C$ is the conclusion of a rule with premise(s) containing

$A_1 \vee A_2$, e.g. $(\vee \mathrm{L})^*$:

$$\frac{A_1 \vee A_2, \Gamma', D \vdash^* C \qquad A_1 \vee A_2, \Gamma', E \vdash^* C}{A_1 \vee A_2, \Gamma', D \vee E \vdash^* C};$$

by induction, we now have $I, \Gamma', D \vdash C$ and $I, \Gamma', E \vdash C$, hence $I, \Gamma', D \vee E \vdash C$ by $(\vee \mathrm{L})$. Similarly for other rules.

(III) $A_0 = A_1 \to A_2$. Let

$$(\to \mathrm{L}_i)^* \quad \frac{\Gamma_i, A_1 \to A_2 \vdash^* A_1 \qquad A_2, \Gamma_i \vdash^* C_i}{\Gamma_i, A_1 \to A_2 \vdash^* C_i}$$

($i = 1, \ldots, n$; $n \geq 0$) be all the instances of $(\to \mathrm{L})^*$ in Π^+ with $A_1 \to A_2$ as main formula. By 2.4 we have Γ_i, $A_1 \to A_2 \vdash A_1$ ($i = 1, \ldots, n$), and with 4.2 we find interpolants J_i ($i = 1, \ldots, n$) in $[\wedge, \vee, \to]$ with

(3) $\qquad \Gamma_i \vdash J_i,$

(4) $\qquad J_i, A_1 \to A_2 \vdash A_1,$

(5) $\qquad p(J_i) \subseteq p(\Gamma_i) \cap p(A_1 \to A_2).$

Now put

$$I(A_1 \to A_2, \Pi^+) := (J_1 \vee \cdots \vee J_n) \Rightarrow I(A_2, \Pi^+).$$

We show (i)–(iv) of (2), writing I for $I(A_1 \to A_2, \Pi^+)$.

(i) holds by Lemma 3.2(ii).

(ii) By (5) and (1) we have $p(J_i) \subseteq p(B) \cap p(A)$ ($i = 1, \ldots, n$), so $p(I) \subseteq p(A) \cap p(B)$ by the induction hypothesis (2) and Lemma 3.2(ii).

(iii) By (4) we have $J_i, A_1 \to A_2 \vdash A_1$ ($i = 1, \ldots, n$), hence $J_1 \vee \cdots \vee J_n$, $A_1 \to A_2 \vdash A_1$; also $A_2 \vdash I(A_2, \Pi^+)$ by the induction hypothesis (2). This gives $A_1 \to A_2 \vdash (J_1 \vee \cdots \vee J_n) \to I(A_2, \Pi)$, so with Lemma 3.2(ii) we get $A_1 \to A_2 \vdash I$.

(iv) Let $A_1 \to A_2, \Gamma \vdash^* C$ be a sequent in Π^+. Three cases:

(a) It is an axiom: then so is $I, \Gamma \vdash C$.

(b) It is the conclusion of one of the $(\to \mathrm{L}_i)^*$, so $\Gamma = \Gamma_i$, $C = C_i$. Now $\Gamma_i \vdash J_i$ by (3), so $\Gamma_i \vdash J_1 \vee \cdots \vee J_n$; also, by induction hypothesis (2), $I(A_2, \Pi^+), \Gamma_i \vdash C_i$, hence we have $(J_1 \vee \cdots \vee J_n) \to I(A_2, \Pi), \Gamma_i \vdash C_i$, i.e. (by Lemma 3.2(ii)) $I, \Gamma_i \vdash C_i$.

(c) It is the conclusion of another instance of a rule, with $A_1 \to A_2$ as main formula: now (iv) follows with induction, as under (IIc) above. □

4.6. Theorem. *Interpolation holds in* $[\vee, \neg]$.

Proof. Analogous to that of Theorem 4.5 above. We only present the differences. Π^+ is defined in 4.5, but now with the exception

(∗) if the rule involved is an instance of $(\neg \mathrm{L})^*$ with a subformula occurrence of A as main formula, then the premise is *not* in Π^+.

This Π^+ satisfies (1) of 4.5, and we prove the following analogue of (2):

(6) *for every strictly positive subformula occurrence A_0 of A, there is an $I = I(A_0, \Pi^+)$ with* (i) $I \in [\vee, \neg]$, (ii) $p(I) \subseteq p(A) \cap p(B)$, (iii) $A_0 \vdash I$, *and* (iv) *for every sequent $A_0, \Gamma \vdash^* C$ ($A_0 \notin \Gamma$) in Π^+ we have $I, \Gamma \vdash C$.*

$A_0 = P$ or $A_0 = A_1 \vee A_2$. As in 4.5.
$A_0 = \neg A_1$. Let

$$(\neg \mathrm{L}_i) \quad \frac{\Gamma_i \vdash A_1}{\Gamma_i, \neg A_1 \vdash C_i}$$

($i = 1, \ldots, n$; $n \geq 0$) be all the instances of ($\neg$L) in Π^+ with $\neg A_1$ as main formula. By 4.2 we find interpolants J_i ($i = 1, \ldots, n$) in [$\wedge, \vee, \neg$] with

$$\Gamma_i \vdash J_i, \tag{7}$$

$$J_i \vdash A_1, \tag{8}$$

$$p(J_i) \subseteq p(\Gamma_i) \cap p(A_1). \tag{9}$$

Now put

$$I(\neg A_1, \Pi^+) := \Downarrow(J_1 \vee \cdots \vee J_n).$$

We show (i)–(iv) of (6), writing I for $I(\neg A_1, \Pi^+)$.

(i) holds by Lemma 3.3(ii).

(ii) By (9) and (1) we have $p(J_i) \subseteq p(B) \cap p(A)$ ($i = 1, \ldots, n$) so $p(I) \subseteq p(A) \cap p(B)$ by the induction hypothesis (6) and Lemma 3.3(ii).

(iii) By (8) we have $J_i \vdash A_1$ ($i = 1, \ldots, n$), hence $J_1 \vee \cdots \vee J_n \vdash A_1$. This gives $\neg A_1 \vdash \neg(J_1 \vee \cdots \vee J_n)$, so with Lemma 3.3(ii) we get $\neg A_1 \vdash I$.

(iv) Let $\neg A_1, \Gamma \vdash^* C$ be a sequent in Π^+. Three cases:

(a) It is an axiom: then so is $I, \Gamma \vdash C$.

(b) It is the conclusion of one of the $(\neg \mathrm{L}_i)^*$, so $\Gamma = \Gamma_i$, $C = C_i$. Now $\Gamma_i \vdash J_i$ by (7), so $\Gamma_i \vdash J_1 \vee \cdots \vee J_n$; hence we have $\neg(J_1 \vee \cdots \vee J_n)$, $\Gamma_i \vdash C_i$, i.e. (by Lemma 3.3(ii)) $I, \Gamma_i \vdash C_i$.

(c) It is the conclusion of an instance of another rule, with $\neg A_1$ as main formula: now (iv) follows by induction, as under (IIc) in 4.5. □

§5. Fragments without $\top$ or $\bot$. To make life simpler, we considered $\top$ and $\bot$ as constants which are present in every fragment. If we do not choose to do so, we have to be slightly more careful, as we shall now explain.

5.1. $\top$ is definable in fragments containing $\to$ (by $P \to P$), $\wedge$ and $\neg$ (by $\neg(P \wedge \neg P)$), or $\vee$ and $\neg$ (by $\neg\neg(P \vee \neg P)$); similarly, $\bot$ is definable in fragments containing $\neg$ and $\wedge$, $\neg$ and $\vee$, or $\neg$ and $\to$. In such fragments we have e.g. the following derivable sequents:

$$P \to P \vdash Q \to Q, \qquad P \wedge \neg P \vdash Q \wedge \neg Q,$$

and an interpolant I for any of these must satisfy $p(I) = \varnothing$, which is impossible without constants. The obvious remedy is to strengthen the premise in the formulation of the interpolation theorem by adding any of the following conclusions: $p(A) \cap p(B) \neq \varnothing$, or $not(A \vdash \bot)$ and $not(\vdash B)$.

§6. Other fragments, open problems. In this last section we consider fragments containing $\leftrightarrow$ and $\neg\neg$, and sketch some attempts to prove interpolation.

6.1. The only fragments based on $\wedge$, $\vee$, $\to$ and $\leftrightarrow$ are [$\leftrightarrow$], [$\wedge, \to, \leftrightarrow$]

($\equiv [\wedge, \rightarrow]$) and $[\wedge, \vee, \rightarrow, \leftrightarrow]$ ($\equiv [\wedge, \vee, \rightarrow]$) for

$$\begin{aligned} A \leftrightarrow (A \rightarrow B) &\equiv A \wedge B, \\ (A \wedge B) \leftrightarrow A &\equiv A \rightarrow B, \\ B \leftrightarrow (A \vee B) &\equiv A \rightarrow B, \\ (A \leftrightarrow B) \leftrightarrow (A \vee B) &\equiv A \wedge B. \end{aligned}$$

So $[\leftrightarrow]$ is the only new fragment. We conjecture that interpolation holds, but a proof has not been found. We sketch two approaches.

6.2. The sequent calculus $\mathbb{SC}(\leftrightarrow)$ for $[\leftrightarrow]$ is defined as the axioms (P), ($\top$) and ($\bot$), plus the following rules:

$$(\leftrightarrow\mathrm{R}) \quad \frac{\Gamma, A \vdash B \qquad \Gamma, B \vdash A}{\Gamma \vdash A \leftrightarrow B},$$

$$(\leftrightarrow\mathrm{L}) \quad \frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \leftrightarrow B \vdash C}, \qquad \frac{\Gamma \vdash B \qquad \Gamma, A \vdash C}{\Gamma, A \leftrightarrow B \vdash C}.$$

To see that a formula $A \in [\leftrightarrow]$ is derivable in $\mathbb{SC}(\leftrightarrow)$ if and only if it is derivable in $\mathbb{SC}$ (considering $A \leftrightarrow B$ as an abbreviation of $(A \rightarrow B) \wedge (B \rightarrow A)$), one argues as follows. Define $\mathbb{SC}^+$ as the union of $\mathbb{SC}$ and $\mathbb{SC}(\leftrightarrow)$, show that the cut rule is a derived rule in $\mathbb{SC}^+$, observe that

$$(A \rightarrow B) \wedge (B \rightarrow A) \vdash A \leftrightarrow B \quad \text{and} \quad A \leftrightarrow B \vdash (A \rightarrow B) \wedge (B \rightarrow A)$$

are derivable in $\mathbb{SC}^+$ and conclude, for sequents $\Gamma \vdash A$ in $[\leftrightarrow]$: $\Gamma \vdash A$ derivable in $\mathbb{SC}(\leftrightarrow)$ if and only if $\Gamma \vdash A$ derivable in $\mathbb{SC}^+$ if and only if $\Gamma \vdash A$ derivable in $\mathbb{SC}$.

It is not immediately clear how to prove interpolation for $[\leftrightarrow]$ with $\mathbb{SC}(\leftrightarrow)$: the "interpolation rules" to be used in Schütte's method are e.g.

$$(\mathrm{i} \leftrightarrow \mathrm{L1}) \quad \frac{\Gamma[I_1]\Delta \vdash A \qquad \Gamma[I_2]B, \Delta \vdash C}{\Gamma[I_1 \wedge I_2]A \leftrightarrow B, \Delta \vdash C},$$

$$(\mathrm{i} \leftrightarrow \mathrm{L2}) \quad \frac{\Delta[I_1]\Gamma \vdash A \qquad \Gamma, B[I_2]\Delta \vdash C}{\Gamma, A \leftrightarrow B[I_1 \rightarrow I_2]\Delta \vdash C},$$

so we get interpolants containing $\wedge$ and $\rightarrow$.

6.3. Another candidate method to prove interpolation for $[\leftrightarrow]$, which works in classical logic (see [Z78]), is: show $A(p) \vdash A(A(\top))$. Unfortunately, this does not hold for all $A \in [\leftrightarrow]$: to see this, take $A(p) := (p \leftrightarrow q) \leftrightarrow (p \leftrightarrow r)$, then $A(A(\top)) \equiv A(q \leftrightarrow r) \equiv q \leftrightarrow r$, but

$$(p \leftrightarrow q) \leftrightarrow (p \leftrightarrow r) \vdash q \leftrightarrow r \text{ is } \textit{not} \text{ derivable;}$$

to see this, take $p := \bot$ and $q := \neg\neg r$. Despite these unsuccessful attempts, we state the following *conjecture*: interpolation holds for $[\leftrightarrow]$.

6.4. *Fragments with* $\neg\neg$. We introduce a new connective $\sim$ for double negation. The sequent calculus $\mathbb{SC}(\sim)$ is based on sequents $\Gamma \vdash A$ or $\Gamma \vdash$ (think of this last sequent as being equivalent to $\Gamma \vdash \bot$), and contains the axioms and rules of $\mathbb{SC}$

(possibly with sequents $\Gamma \vdash$), together with

$$(\neg \mathrm{L})\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash}, \qquad (\neg \mathrm{R})\ \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A},$$

$$(\sim \mathrm{L})\ \frac{\Gamma, A \vdash}{\Gamma, \sim A \vdash}, \qquad (\sim \mathrm{R})\ \frac{\Gamma, \neg A \vdash}{\Gamma \vdash \sim A},$$

$$(\mathrm{W})\ \frac{\Gamma \vdash}{\Gamma \vdash A}.$$

The cut rule is a derived rule of $\mathbb{SC}(\sim)$, and it is easy to see that

$$\sim A \vdash \neg\neg A \quad \text{and} \quad \neg\neg A \vdash \sim A$$

are derivable in $\mathbb{SC}(\sim)$. For fragments containing $\sim$ but not $\neg$, the rule ($\neg$R) can be skipped, but ($\neg$L) is still needed because of ($\sim$R). To extend Schütte's method, the following "interpolation rules" are needed:

$$(\mathrm{i} \neg \mathrm{L1})\ \frac{\Gamma[I]\Delta \vdash A}{\Gamma[I]\neg A, \Delta \vdash}, \qquad (\mathrm{i} \neg \mathrm{L2})\ \frac{\Delta[I]\Gamma \vdash A}{\Gamma, \neg A[\neg I]\Delta \vdash},$$

$$(\mathrm{i} \sim \mathrm{L1})\ \frac{\Gamma[I]\Delta, A \vdash}{\Gamma[I]\Delta, \sim A \vdash}, \qquad (\mathrm{i} \sim \mathrm{L2})\ \frac{\Gamma, A[I]\Delta \vdash}{\Gamma, \sim A[\sim I]\Delta \vdash}.$$

Unfortunately, this extension of Schütte's method may (by (i$\neg$L2)) introduce $\neg$ in the definition of an interpolant for $A \vdash B$ with A and B in some fragment containing $\sim$, but not $\neg$. Closer inspection learns that (i$\neg$L2) is only needed in fragments containing $\neg$ or $\rightarrow$, so interpolation holds in the rather trivial fragments $[\sim]$, $[\sim, \wedge]$, $[\sim, \vee]$ and $[\sim, \wedge, \vee]$. For the other fragments, the question arises: which fragments containing $\sim$ and $\rightarrow$ satisfy interpolation?

6.5. *Uniform interpolation.* Finally we state the following open problem: does IpC satisfy *uniform* interpolation, i.e. are there, for every formula A and every $\mathbf{P} \subseteq a(A)$, a uniform right interpolant $I_{\mathrm{R}} = I_{\mathrm{R}}(A, \mathbf{P})$ and a uniform left interpolant $I_{\mathrm{L}} = I_{\mathrm{L}}(A, \mathbf{P})$ such that

$A \vdash I_{\mathrm{R}}$ and $I_{\mathrm{L}} \vdash A$,

$a(I_{\mathrm{R}}), a(I_{\mathrm{L}}) \subseteq \mathbf{P}$,

for all B with $A \vdash B$ and $a(A) \cap a(B) \subseteq \mathbf{P}$ we have $I_{\mathrm{R}} \vdash B$, and

for all B with $B \vdash A$ and $a(A) \cap a(B) \subseteq \mathbf{P}$ we have $B \vdash I_{\mathrm{L}}$?

In classical logic, the left and right variant are equivalent. Uniform interpolation holds for classical propositional logic, but not for classical predicate logic (see [H63]), and hence not for intuitionistic predicate logic.

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